MA3025 Solutions Exam # 2

Due 9am November 15**th,** 2007 **Name** ______________________________

Instructor: Dr. Ralucca Gera

Show all necessary work in each problem to receive credit. Please turn in well-organized work and complete solutions. You may ONLY use your notes and Rosen book (no collaboration is allowed either).

1. (10 points) True or false (no need to justify):

(a) Every 2×2 matrix with a nonzero determinant has an inverse.
Solution: True.

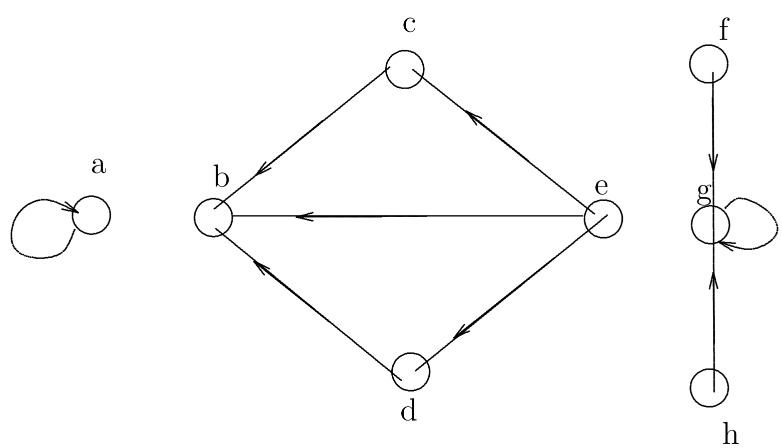
(b) The recurrence $a_n = 2a_{n-1} + \sqrt{2}a_{n-2} + \pi a_{n-3}$ with $a_0 = 0$, $a_1 = 2$ and $a_2 = 3$ is a linear homogeneous recurrence with constant coefficients of degree 3.
Solution: True

(c) The equation $a_n = (n-1)!, n \geq 1$ is a solution to the recurrence $a_n = n \cdot a_{n-1}, n \geq 2$ with $a_1 = 1$.
Solution: False

(d) Is the relation given by the following matrix symmetric? $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Solution: No because $a_{4,1} \in M_R$ but $a_{1,4} \notin M_R$

(e) Is the relation given by the following digraph transitive?



Solution: Yes.

2. (15 points) Recall that the Fibonacci sequence F_n is defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence L_n is similarly defined, by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. (The two sequences use the same recurrence, but with different initial conditions.) Prove that, for all $n \geq 2$ we have that $5F_{n+2} = L_{n+4} - L_n$.

Proof: We prove $P(n) : 5F_{n+2} = L_{n+4} - L_n$ for $n \geq 2$ by induction.
Basis Step: $P(2) : 5F_4 = 5 \cdot 3 = 15$ and $L_6 - L_2 = 18 - 3 = 15$.
$P(3) : 5F_5 = 5 \cdot 5 = 25$ and $L_7 - L_3 = 29 - 4 = 25$.

Inductive step: Assume $P(2) \wedge P(3) \wedge \ldots \wedge P(k-1) \wedge P(k)$ and prove
$P(k+1) : 5F_{k+3} = L_{k+5} - L_{k+1}$. Note that

$$5F_{k+3} = 5(F_{k+2} + F_{k+1}) = 5F_{k+2} + 5F_{k+1} = L_{k+4} - L_k + L_{k+3} - L_{k-1} = L_{k+5} - L_{k+1}.$$

■

3. (15 points) Let L_n be defined by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Prove that for nonengative integers n we have that $\sum_{i=0}^{n} L_i = L_{n+2} - 1$.

Proof: We prove $P(n) : \sum_{k=0}^{n} L_k = L_{n+2} - 1$, $n \geq 0$.

For a basis, we consider $P(0) : \sum_{k=0}^{0} L_k = L_0 = 2$ and on the other hand $L_{0+2} - 1 = 3 - 1 = 2$. Thus the basis case is true.

Assume that $P(k)$ is true, and prove $P(k+1)$. Observe that

$$\sum_{i=0}^{k+1} L_i = \sum_{i=0}^{k} L_i + L_{k+1} = L_{k+2} - 1 + L_{k+1} = L_{k+3} - 1,$$

which is what we needed to show. ■

4. (10 points) Solve $a_n = \frac{a_{n-2}}{9}$ for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$

Solution: The characteristic equation is $r^n = \frac{r^{n-2}}{9}$ or $r^2 = 1/9$. And so $r = \pm\frac{1}{3}$. Thus

$$a_n = \alpha\Big(\frac{1}{3}\Big)^n + \beta\Big(-\frac{1}{3}\Big)^n$$

Now $n = 0 \quad \rightarrow \quad 0 = \alpha + \beta$
and $n = 1 \quad \rightarrow \quad 1 = \alpha(1/3) + \beta(-1/3)$.
And so $\alpha = -\beta$, which implies that $1 = \frac{2\alpha}{3}$. Thus $\alpha = \frac{3}{2}$ and $\beta = -\frac{3}{2}$. Therefore for $n \geq 0$ we have that

$$a_n = \frac{3}{2}\Big(\frac{1}{3}\Big)^n - \frac{3}{2}\Big(-\frac{1}{3}\Big)^n, n \geq 0.$$

5. (10 points) Find a recurrence for the relation $a_n = (-1)^n n!$ for $n \geq 0$. Simplify as much as possible.

Solution: Method 1: Note that
$a_n - a_{n-1} = (-1)^n n! - (-1)^{n-1}(n-1)! = (-1)^{n-1}(n-1)!\Big(-n-1\Big) = (-1)^{n-1}(n-1)!(-1)(n+1) = (-1)^n(n-1)!(n+1)$. And so $a_n = a_{n-1} + (-1)^n(n-1)!(n+1), n \geq 1$ with $a_0 = 1$.

Method 2: $\frac{a_n}{a_{n-1}} = \frac{(-1)^n n!}{(-1)^{n-1}(n-1)!} = -n$. And so $a_n = -na_{n-1}, n \geq 1$ with $a_0 = 1$.

6. (10 points)

(a) Find the number of terms in the formula for the number of elements in the union of 4 sets given by the principle of inclusion-exclusion. Some terms may be zero, and you shoud count them as well.

Solution: Method 1: The number of terms is given by $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4} = 2^4 - 1 = 15$ using the binomial theorem (or the row of Pascal's triangle with $n = 4$).

Method 2: Let A_1, A_2, A_3, and A_4 be the four sets. Then $|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$. So there are 15 terms, each one of them corresponding to a subset of 1 or 2 or 3 or 4 elements. Thus as mentioned in method 1, 15 is the total number of subsets of the set $\{A_1, A_2, \ldots A_4\}$ except the empty set (so we have $2^4 - 1$ terms)

(b) How many bit strings of length 14 do not contain 12 consecutive 1s?

Solution: Let A be the event that has 12 consecutive ones as the first 12 bits, B be the event that has 12 consecutive ones as middle 12 bits, and C be the event that has 12 consecutive ones as the last 12 bits. Then the total number of bit strings of length 14 do contain 12 consecutive 1s is given by

$$|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C| = 4+4+4-2-2-1+1 = 8.$$

Therefore the total number of bit strings of length 14 do NOT contain 12 consecutive 1s is given by $2^{14} - 8 = 16376$.

2 extra credit points Find the number of terms in the formula for the number of elements in the union of 40 sets given by the principle of inclusion-exclusion. Some terms may be zero, and you should count them as well.
Solution: The number of terms is given by $\binom{40}{1}+\binom{40}{2}+\binom{40}{3}+\ldots+\binom{40}{40} = \sum_{i=1}^{40} = 2^{40}-1$ using the binomial theorem (or the row of Pascal's triangle with $n = 40$).

7. (10 points) Let $A = \{-1, 0, 1, 2, 3, 4\}$ and $R = \{(-1, 2), (-1, 3), (0, 0), (1, 1), (2, 3)\}$.

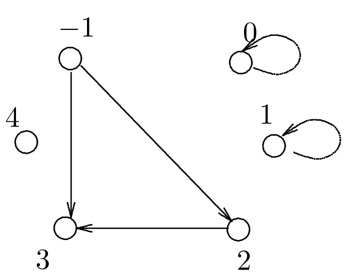
(a) Find R^2 and R^3
Then $R^2 = \{(-1, 3), (0, 0), (1, 1)\}$ as a composition of elements of R.
Also $R^3 = \{(0, 0), (1, 1)\}$ as a composition of an element of R and an element of R^2.

(b) is the element $(1, 3)$ in R^{2007}?
No. $(1, 3) \notin R^{2007}$ since R is transitive and so $R^n \subseteq R$, for all $n \geq 1$, so in particular for $n = 2007$, and also $(1, 3) \notin R$.

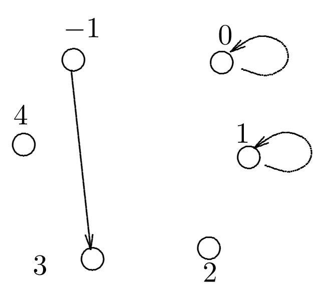
(c) list each of R, R^2 and R^3 with a matrix. $M_R = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

$$M_{R^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ M_{R^3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

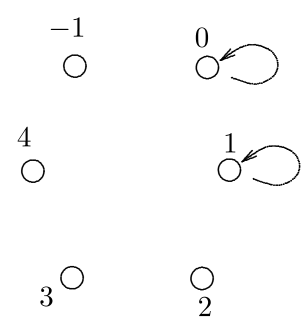
(d) draw the directed graphs that represent R, R^2 and R^3.



M_R



M_{R^2} :



M_{R^3} :

8. (20 points) let A be the set of all binary strings of length 100. Define a relation R on A by $(x, y) \in R$ if the binary strings x and y agree in the first and the last bit. Answer with explanations if R:

(a) reflexive?
Yes, since x agrees with itself in the first and last bit, for $\forall x \in A$

(b) irreflexive?
No, since it is reflexive.

(c) symmetric?
Yes. If x and y agree in the first and the last bit, so do y and x

(d) antisymmetric?
No. Counterexample: The binary strings of length 100, say $x = 00\ldots0$ and $y = 0111\ldots1110$ satisfy $(x, y) \in R$ and $(y, x) \in R$, yet $x \neq y$.

(e) transitive? Yes: If $(x, y) \in R$ and $(y, z) \in R$, then both x and y begin and end with the same bit, and also y and z begin and end with the same bit. Since y is common, it follows that x and z begin and end with the same bit, and so $(x, z) \in R$.

(f) Find the equivalence classes if they exist.
The equivalence classes are:
[0...0] = the class of all binary bits of length 100 that begin and end with 0.
[0...1] = the class of all binary bits of length 100 that begin with 0 and end with 1.
[1...0] = the class of all binary bits of length 100 that begin with 1 and end with 0.
[1...1] = the class of all binary bits of length 100 that begin and end with 1.

(g) How many elements are there in each of the equivalence classes above?
Each class has 2^{98} elements

(h) Do the classes form a partion? If so, what are they a partition of? If not, what set should they partition?
Yes, the classes form a partition of A. That is $A = [0...0] \cup [0...1] \cup [1...0] \cup [1...1]$

(i) How many elements are there in the relation R above?
One can form $(2^{98})^2$ elements in R from each of the classes, since for each class each of the 2^{98} elements can be paired up with each of the other 2^{98} elements. And so R has $4 \cdot (2^{98})^2 = 2^2 \cdot 2^{196} = 2^{198}$ elements.

(EXTRA CREDIT: 5 points) Let R be a relation defined on the set of integers by the division property:

$$R = \{(a, b) : a|b\}$$

Is R:

(a) reflexive?
No, since $0 \not| 0$ so $(0, 0) \notin R$, yet $0 \in \mathbb{Z}$

(b) symmetric?
No. Counterexample: $(2, 6) \in R$ since $2|6$, yet $(6, 2) \notin R$ since $6 \not| 2$.

(c) antisymmetric?
No. Counterexample: $(-3, 3) \in R$ and $(3, -3) \in R$, but $-3 \neq 3$. (NOTE THAT: If $(x, y) \in R$ then $x|y$. Also if $(y, x) \in R$, then $y|x$. Since $x|y$ and $y|x$, we have that $x = y$ or $x = -y$ (See #5 page 208).)

(d) transitive? Yes: If $(x, y) \in R$ and $(y, z) \in R$, then $x|y$ and $y|z$. And so $x|z$, implying that $(x, z) \in R$.

(e) Find the equivalence classes if they exist.
The equivalence classes don't exist since the relation is not symmetric.